Fermion Zero Modes and Black-Hole Hypermultiplet with Rigid Supersymmetry

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Abstract

The gravitini zero modes riding on top of the extreme Reissner–Nordström black-hole solution of $N = 2$ supergravity are shown to be normalizable. The gravitini and dilatini zero modes of axion-dilaton extreme black-hole solutions of $N = 4$ supergravity are also given and found to have finite norms. These norms are duality invariant. The finiteness and positivity of the norms in both cases

are found to be correlated with the Witten–Israel–Nester construction; however, we have replaced the Witten condition by the pure-spin-$\frac{3}{2}$ constraint on the gravitini. We compare our calculation of the norms with the calculations which provide the moduli space metric for extreme black holes.

The action of the $N = 2$ hypermultiplet with an off-shell central charge describes the solitons of $N = 2$ supergravity. This action, in the Majumdar–Papapetrou multi-black-hole background, is shown to be $N = 2$ rigidly supersymmetric.

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1   Introduction

Classical solutions of supergravity theories with unbroken supersymmetries have attracted much attention in recent years. This is due to the many interesting properties that they usually share: they minimize the energy for given values of the charges (they saturate supersymmetric Bogomol'nyi bounds [1]) and so they are stable and can be considered solitons [2], alternative vacuums or ground states of the theory (depending on the number of unbroken supersymmetries). In general, multi-center solutions describing an arbitrary number of these solitons in equilibrium exist. In some cases they also enjoy non-renormalization theorems [3].

The supersymmetric solutions usually considered have vanishing fermionic fields. It is, however, easy to obtain new solutions with non-vanishing fermion fields starting with the purely bosonic ones and performing supersymmetry transformations. If the supersymmetry parameters used vanish asymptotically (*i.e.* the supersymmetry transformations are *trivial*), the new solutions obtained in this way are gauge-equivalent to the original ones. If the supersymmetry parameters converge asymptotically to global supersymmetry parameters then the new solutions obtained are no longer gauge-equivalent to the original ones (it is not possible to go back to them by using asymptotically vanishing supersymmetry parameters). In this way, if non-trivial supersymmetry parameters with the right regularity properties exist, one can generate a whole supermultiplet of solutions [4]. This program has been successfully carried out for the Majumdar–Papapetrou solutions of $N = 2$ supergravity in a series of papers by Aichelburg, *et al.* [5].

From the point of view of the representation theory of the supersymmetry algebra [6], the supermultiplets of solutions generated in this way are *shortened* supermultiplets whose dimension is smaller than the dimension of the original supersymmetry multiplet. This is so because, by definition, there are some non-trivial supersymmetry transformations (those generated by the Killing spinors) that leave the original solution invariant. Only the non-trivial supersymmetry parameters corresponding to the broken supersymmetries (that we will call "anti-Killing spinors") generate new non-gauge equivalent solutions.

Having found the supermultiplet structure of these supersymmetric solitons, it is natural to look for a theory describing its dynamics. This would be the supersymmetric quantum field theory of the solitons of the original theory. There are well-known examples of quantum field theories which describe the quantum relativistic dynamics of the solitons of another theory. The most famous example of this *duality* is the relation between the Thirring model and the sine-Gordon model. The former has as elementary excitations the solitons of the second one [7]. It was argued by Montonen and Olive [8] that there should exist quantum field theories of the magnetic monopoles of known gauge theories. An example along the lines of this conjecture was pointed out by Osborn in Ref. [9]. He showed that the spectrum of solitons of $d = 4$, $N = 4$ super-Yang–Mills, which have two unbroken supersymmetries, is the same as the spectrum of elementary excitations of the original theory and suggested that the theory could be self-dual.

Although the Montonen–Olive conjecture does not directly apply to the kind of solitons that we will be concerned with here (extreme black holes), it is possible that a quantum field theory describing the dynamics of these objects exists. The supermultiplet structure determines, to a

large extent, the form of the theory. In particular, since black holes would correspond to massive states, massive representations of the supersymmetry algebra are needed. The supergravity multiplets one starts with are massless, and, therefore, they cannot describe the supermultiplets of black-hole solutions.

Another property of the purely bosonic supersymmetric solutions is that they admit fermion zero modes, *i.e.* classical solutions of the fermion (Dirac or Rarita–Schwinger) equations of motion in that background. Furthermore, it is straightforward to generate these fermion zero modes: an infinitesimal supersymmetry transformation generated by anti-Killing spinors generates non-trivial fermion fields which solve the equations of motion and do not change the bosonic background. The fermion zero modes are thus the lowest order contribution to the fermion fields in each solution in the supermultiplet.

So far, the normalizability of these fermion zero modes (which is an important issue if the supermultiplet of solutions is going to be interpreted as a supermultiplet of states) had always been implicitly assumed. It has recently been shown in Ref. [10] that this is not always so. This raises the question as to whether in the previously known cases the norms of the fermion zero modes (which never were explicitly calculated) are finite or not, and why.

In this paper we will demonstrate the existence of normalizable fermion zero modes in the extreme Reissner–Nordström (ERN) black-hole background and in the extreme dilaton black-hole (EDBH) background. We will show that the normalizability of these zero modes can be understood in terms of the saturation of the Bogomol'nyi positivity bounds for the ADM mass by using the four–dimensional $N = 2$ and $N = 4$ versions of the Witten–Israel–Nester (WIN) [11] construction presented in Refs. [12] and [13], respectively.

In addition, we will investigate the action of the massive $N = 2$ hypermultiplet whose supermultiplet structure corresponds to that of the ERN multiplet and it is therefore *the* candidate to describe the dynamics of the ERN black holes. We will see that the existence of fermionic zero modes in the ERN black-hole background leads to the existence of rigidly (as opposed to globally) supersymmetric theories. In particular, we will see that the $N = 2$ hypermultiplet may be placed on an ERN background. We expect that the techniques used here may be applicable to other curved geometries.

2   Normalizability of the ERN zero modes

To begin, we recall some well-known facts. We start by describing the Majumdar–Papapetrou (MP) solutions of the Einstein–Maxwell theory [14] for zero magnetic and positive electric charges[3],

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[3]Our metric's signature is $(+ - --)$, the gamma matrices are those of Ref. [15] and in particular satisfy $\{\gamma^\mu, \gamma^\nu\} = +2g^{\mu\nu}$, where γ_0 is Hermitean, the γ_i's are anti-Hermitean and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Greek indices are curved, the first Latin alphabet letters $a, b, c, \ldots$ are flat indices and the indices $i, j, k, \ldots$ run from 1 to 3. Underlined indices ($\underline{0}, \underline{i}$, etc.) are always curved. $\epsilon^{\underline{0123}} = +(-g)^{-\frac{1}{2}}$.

$$ds^2 = V^{-2}dt^2 - V^2 d\vec{x}^2 \ , \qquad A = -\frac{1}{\kappa}V^{-1}\, dt \ , \tag{1}$$

where $\kappa^2 = 4\pi G$ and the function V does not depend on time and satisfies

$$\partial_{\underline{i}}\partial_{\underline{i}}V = 0\,. \tag{2}$$

The requirements of asymptotic flatness and regularity determine it to be of the form[4]

$$V(\vec{x}) = 1 + \sum_s \frac{GM_s}{|\vec{x} - \vec{x}_s|}\,, \tag{3}$$

where the horizon of the sth black hole is located at $\vec{x} = \vec{x}_s$ and its electric charge is $Q_s = G\kappa^{-1}M_s$. The parameter M_s is usually interpreted as the mass of the sth black hole. However, in this space-time, there is no way to calculate the mass of each individual black hole since there is only one asymptotically flat region, common for all black holes. Thus, strictly speaking, one can only say that the ADM mass of this space-time is $M_{ADM} = \sum_s M_s$. Nevertheless, taking into account that this background describes charged black holes in static equilibrium which share many of the characteristics of ERN black holes, it is natural to identify M_s with the mass of the sth black hole.

Since the multiplet of $d = 4, N = 2$ supergravity [16, 17] is $(e_\mu{}^a, A_\mu, \psi_\mu)$, where A_μ is a $U(1)$ gauge field and ψ_μ is a complex spin-$\frac{3}{2}$ field (*i.e.* it is a Dirac spinor for each value of the vector index μ), it is clear that the Einstein–Maxwell theory can be embedded in it, and the MP solutions can be considered solutions of $d = 4$, $N = 2$ supergravity with the only fermion field of the theory vanishing: $\psi_\mu = 0$.

A remarkable feature of this background is that it admits $N = 2$ supergravity Killing spinors [12, 18], *i.e.* a solution of the equation

$$\hat{\nabla}_\mu \epsilon = 0 \ . \tag{4}$$

Here $\hat{\nabla}_\mu$ is the $N = 2$ supercovariant derivative given in terms of the gravitational covariant derivative, ∇_μ by

$$\hat{\nabla}_\mu \equiv \nabla_\mu - \frac{1}{4}\kappa \not{F}\gamma_\mu \ , \tag{5}$$

where $\not{F} = \gamma^{\mu\nu}F_{\mu\nu}$ and $F_{\mu\nu}$ is the field–strength of the gauge field A_μ and ϵ is a Dirac spinor. That solution is given by

$$\epsilon_{(k)} = V^{-1/2}\mathcal{C}_{(k)} \ , \tag{6}$$

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[4]Dropping the condition of asymptotic flatness other solutions are possible. Remarkably enough, if one deletes the 1 in Eq. (3) we get, for a single $\vec{x}_s$, Robertson–Bertotti's solution.

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where $\mathcal{C}_{(k)}$ is a constant spinor satisfying the condition

$$\gamma_0 \, \mathcal{C}_{(k)} = + \, \mathcal{C}_{(k)} \,, \tag{7}$$

and is given in terms of a complex two-component spinor, c, by $\mathcal{C}_{(k)}{}^t = (c^t, c^t)$. This means that the background given above has one unbroken supersymmetry in $N = 2$ supergravity.

If we perform an infinitesimal supersymmetry transformation with a supersymmetry parameter ϵ that does not vanish asymptotically and that it is not a Killing spinor either (we will call it an "anti-Killing" spinor and denote it by $\epsilon^{(\bar{k})}$), the bosonic fields (the metric and vector field) will remain invariant but a non-trivial fermionic field that solves the gravitini field equations in this background will be generated. This fermionic zero-mode is therefore given by [5]

$$\psi_\mu \;=\; \frac{1}{\kappa} \hat{\nabla}_\mu \epsilon^{(\bar{k})} \;\;, \tag{8}$$

where $\epsilon^{(\bar{k})}$ is the anti-Killing spinor. By construction it satisfies the covariant Dirac equation

$$\gamma^\mu \hat{\nabla}_\mu \epsilon^{(\bar{k})} \;=\; \gamma^\mu \nabla_\mu \epsilon^{(\bar{k})} \;=\; 0 \;\;, \tag{9}$$

which implies that the gravitini constructed in this way is always in the gauge in which

$$\gamma^\mu \psi_\mu \;=\; 0 \,. \tag{10}$$

In our case the anti-Killing spinor can be chosen to be, in terms of the Killing spinor,

$$\epsilon^{(\bar{k})} = -i\gamma_5 \epsilon_{(k)} \,, \tag{11}$$

so the explicit expression for the gravitino zero-mode is

$$\psi \;=\; \frac{1}{\kappa} V^{-7/2} \partial_{\underline{i}} V \gamma_i \mathcal{C}^{(\bar{k})} \; dt \;+\; \frac{1}{\kappa} V^{-3/2} \partial_{\underline{j}} V \gamma_j \gamma_i \mathcal{C}^{(\bar{k})} \; dx^{\underline{i}} \;\;,$$

$$\mathcal{C}^{(\bar{k})} \;=\; \begin{pmatrix} c \\ -c \end{pmatrix} \,. \tag{12}$$

Observe that the property of the Killing spinor given in Eq. (7) implies that

$$\gamma_0 \, \mathcal{C}^{(\bar{k})} = - \, \mathcal{C}^{(\bar{k})} \,. \tag{13}$$

If we performed a finite supersymmetry transformation generated by the anti-Killing spinor $\epsilon^{(\bar{k})}$, Eq. (12) would be the lowest order, in $\epsilon^{(\bar{k})}$, contribution to the gravitino. The normalizability of the zero modes is then related to the question of whether a (normalizable) supermultiplet of solutions can be built starting from the MP solutions.

The norm of the gravitino is, by definition

$$\|\psi\|^2 \equiv \int_\Sigma d^3x \sqrt{-g_{(3)}}\, \psi_\mu{}^\dagger \psi_\nu g^{\mu\nu} \ , \tag{14}$$

where Σ is a space-like hypersurface and $g_{(3)}$ is the determinant of the induced metric on it. In our case, Σ will be any constant-time hypersurface and $g_{(3)} = \det g_{ik}$. The norm of the zero-mode of Eq. (12) is

$$\|\psi\|^2 = \frac{1}{2\pi G} \bar{\mathcal{C}}^{(\bar{k})} \mathcal{C}^{(\bar{k})} \int_D d^3x \ V^{-2} (\partial_{\underline{i}} V)^2 \ . \tag{15}$$

where the integration domain D is a subset of the three-dimensional $\mathbf{R}^3$ to be determined later. Let us now specialize to the single ERN black-hole background. In this case D is $\mathbf{R}^3$ with the origin removed ($\mathbf{R}^3 - \{\vec{0}\}$) and we find by a direct calculation of the volume integral that[5]

$$\|\psi\|^2 = 2M\bar{\mathcal{C}}^{(\bar{k})} \mathcal{C}^{(\bar{k})} = 4M\|c\|^2 \ , \tag{16}$$

where $\|c\|^2$ is the norm of the complex, two-component constant spinor. Thus the spin-$\frac{3}{2}$ zero-mode in the ERN black hole background is normalizable.

In order to extend this result to the multi-black-hole case, we first observe that the integrand of the norm is well behaved everywhere (including the origin of $\mathbf{R}^3$, which corresponds to the horizon):

$$\int_D d^3x \ V^{-2}(\partial_{\underline{i}}V)^2 = 4\pi G^2 M^2 \int_0^\infty dr \frac{1}{(r + GM)^2} \ , \tag{17}$$

so we do not expect singular contributions to the integral. In fact, we could have used Gauss' theorem to evaluate the norm as a surface integral at infinity in $\mathbf{R}^3$. To do this, we first rewrite the integrand

$$V^{-2}(\partial_{\underline{i}}V)^2 = -\partial_{\underline{i}}(V^{-1}\partial_{\underline{i}}V) + V^{-1}\partial_{\underline{i}}\partial_{\underline{i}}V \ . \tag{18}$$

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[5]As a matter of fact, we are integrating over a constant time–slice of the ERN geometry. It is well known that these hypersurfaces become bottomless tubes when one approaches the horizon, which is at infinite proper distance over this hypersurface and therefore is not even included in it.

The second term does not contribute to the integral because it vanishes everywhere outside the horizon (Eqs. (2,3)) and on the horizon it appears multiplied by a factor which vanishes there. We get

$$\int_D d^3x \ V^{-2}(\partial_{\underline{i}} V)^2 \ = \ -\int_D d^3x \ \partial_{\underline{i}}(V^{-1}\partial_{\underline{i}} V) \ = \ -\int_{\partial D} dS^{\underline{i}} \ V^{-1}\partial_{\underline{i}} V \,, \tag{19}$$

where we have applied Gauss' theorem and, in the last expression the index $\underline{i}$ is a covariant index in Σ so we can perform the surface integral, in particular, in spherical coordinates. Accordingly, the boundary consists of two disconnected pieces, one at infinity (S^2_∞) and the origin (the horizon). The surface integral at the horizon vanishes because the integrand vanishes there (no singularities there) and we have

$$\int_D d^3x \ V^{-2}(\partial_{\underline{i}} V)^2 \ = \ -\int_{S^2_\infty} dS^{\underline{i}} \ V^{-1}\partial_{\underline{i}} V \,. \tag{20}$$

(Alternatively, knowing that there are no singular contribution to the integral there, we could have included the origin in D and $\partial D = \partial \mathbf{R}^3 = S^2_\infty$, getting the same final result.) The resulting surface integral is easy to calculate and gives the expected result.

Now it is clear that exactly the same arguments go through in the multi-black-hole case ($D = \mathbf{R}^3 - \{\vec{x}_a\}$), where the volume integral becomes too complicated, and we obtain a surface integral which is asymptotically identical to the one in the single-black-hole case. To summarize, for any number of black holes, the norm of the gravitini zero modes is given by

$$\|\psi\|^2 \ = \ -\frac{1}{2\pi G}\bar{\mathcal{C}}^{(\bar{k})}\mathcal{C}^{(\bar{k})}\int_{S^2_\infty} dS^{\underline{i}} V^{-1}\partial_{\underline{i}} V \ = \ 4M_{ADM}\|c\|^2 \,, \tag{21}$$

where $M_{ADM} = \sum_s M_s$ is the total ADM mass.

We would like to understand what is the underlying reason for the normalizability of the gravitini zero modes, since it seems to be just "pure luck" that they are normalizable in some cases and not normalizable in others [10]. Before proceeding we note that the fact that the norm can rewritten as a surface integral over S^2_∞ is suggestive of an ADM construction.

The first crucial observation is that the norm of the gravitino zero-mode, as defined in Eq. (14) is the starting point in the $N = 2$ generalization [12] of the WIN construction [11]. Indeed, since the gravitino is obtained by a supersymmetry transformation as in Eq. (8) with a spinor ϵ, we can proceed as follows. First we define

$$I \ \equiv \ \frac{1}{\kappa^2}\int_\Sigma d\Sigma_\mu \ \overline{\hat{\nabla}_\nu \epsilon} \ \gamma^{\mu\nu\rho}\hat{\nabla}_\rho \epsilon \ . \tag{22}$$

Then using Eq. (8) we see that for our gravitini,

$$I \ = \ \int_\Sigma d\Sigma_\mu \ \bar{\psi}_\nu \gamma^{\mu\nu\rho}\psi_\rho \ . \tag{23}$$

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Next, it follows that

$$\int_{\Sigma} d\Sigma_\mu \ \bar{\psi}_\nu \gamma^{\mu\nu\rho} \psi_\rho \ = \ \int_{\Sigma} d^3x \sqrt{-g_{(3)}} \ \psi_j{}^\dagger \gamma^0 \gamma^{0jk} \psi_k \ ,$$

$$= \ \int_{\Sigma} d^3x \sqrt{-g_{(3)}} \ \psi_\mu^\dagger \psi_\nu g^{\mu\nu} \ = \ \|\psi\|^2 \ . \tag{24}$$

It is noteworthy that in deriving this relation for the gravitino norm, we did not use the Witten condition, $\gamma^{\underline{i}} \hat{\nabla}_{\underline{i}} \epsilon = 0$, which is standard in the WIN construction when deriving the Bogomol'nyi bound. Instead, we have imposed the condition that the gravitino is a pure spin-$\frac{3}{2}$ field: namely,

$$\gamma^\mu \psi_\mu \ = \ 0 \quad \Longrightarrow \quad \gamma^\mu \hat{\nabla}_\mu \epsilon \ = \ 0 \ , \tag{25}$$

Observe that for the MP configuration these equations imply

$$\gamma^{\underline{i}} \psi_{\underline{i}} \ \neq 0 \quad \Rightarrow \quad \gamma^{\underline{i}} \hat{\nabla}_{\underline{i}} \epsilon \ \neq \ 0 \ . \tag{26}$$

Continuing with our analysis, we see that the integral I in Eq. (23), for any spinor ϵ that asymptotically approaches the constant spinor $\mathcal{C}$ ($\epsilon = \mathcal{C} + O(1/r)$) is equal to [12]

$$I \ = \ \int_{\Sigma} d\Sigma_\alpha \ \overline{\mathcal{C}} \left[T^{(matter)\,\alpha}{}_\beta \gamma^\beta + \kappa G^{-1} \left(J^\alpha + i\gamma_5 \tilde{J}^\alpha \right) \right] \mathcal{C} - \overline{\mathcal{C}} \left[-P_\lambda \gamma^\lambda + \kappa G^{-1}(Q + i\gamma_5 P) \right] \mathcal{C} . \tag{27}$$

Let us now specialize these expressions to the case at hand, $\epsilon = \epsilon^{(\bar{k})}$, zero magnetic charge, etc. To begin with, the first term in Eq. (27) is identically zero, since all the sources vanish outside the horizon. Secondly, one can check that the only non-vanishing component of the Lorentz vector $\overline{\mathcal{C}}^{\bar{k}} \gamma^a \mathcal{C}^{\bar{k}}$, has $a = 0$. Finally, given the property (13) and upon using the relation $Q = G\kappa^{-1} \sum_s M_s$ between the total charge and the ADM mass M_{ADM} of the MP solutions, Eq. (27) yields :

$$I \ = \ \left(P^0 + \kappa G^{-1} Q \right) (\mathcal{C}^{\bar{k}})^\dagger \mathcal{C}^{\bar{k}} = 2M_{ADM} (\mathcal{C}^{\bar{k}})^\dagger \mathcal{C}^{\bar{k}} . \tag{28}$$

Hence we see that the results of Ref. [12] lead us directly to the norm of the gravitino. Our explicit evaluation, c.f. Eqns. (14-16), of the norm is to be viewed as a verification of this result.

Now we would like to extend our results to magnetically charged black holes and dyons which can be obtained by electric-magnetic duality rotations of the electromagnetic field strength $F^{\mu\nu}$ in the Einstein–Maxwell theory. The generalization of this symmetry of the equations of motion of the Einstein–Maxwell theory to $N = 2$ supergravity is known from the early days of the theory [17] and is the so-called "chiral-dual" symmetry. The finite chiral-dual transformations of the gravitino and the supercovariant electromagnetic tensor $\hat{F}^{\mu\nu}$ are

$$\psi'_\mu = e^{\frac{i}{2}\theta\gamma^5}\psi_\mu\,, \qquad \hat{F}^{\pm\prime}_{\mu\nu} = e^{\pm i\theta}\hat{F}^\pm_{\mu\nu}\,, \tag{29}$$

where $\hat{F}^\pm_{\mu\nu}$ is the (anti-) self-dual part of $\hat{F}$:

$$\hat{F}^\pm_{\mu\nu} = \tfrac{1}{2}\left(\delta^{\rho\sigma}_{\mu\nu} \pm \tfrac{i}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\right)\hat{F}_{\rho\sigma}\,. \tag{30}$$

We have calculated the gravitino norm for the electrically charged black hole. Instead of performing the new calculations for the magnetic one or for the electromagnetic one, we have only to use the symmetry of the norm (14) under chiral-dual rotation of the vector field and gravitino:

$$\left(\psi_\mu{}^\dagger\psi_\nu\right)' = \psi_\mu{}^\dagger\psi_\nu\,. \tag{31}$$

Alternatively one could consider the transformation rule of $N = 2$ Killing spinors under duality:

$$\epsilon'_{(k)} = e^{\frac{i}{2}\theta\gamma^5}\epsilon_{(k)}\,, \tag{32}$$

which implies that our anti-Killing spinors (11) and gravitino zero modes transform in the same way. Once again, the norm of the gravitino is duality invariant.

3   Normalizability of the dilaton black-hole zero modes

It is interesting to see whether the same happens in other cases. The simplest extension is the purely electric, extreme dilaton black holes (EDBH) [19] which have two unbroken supersymmetries when embedded in $d = 4$, $N = 4$ supergravity [20]. The fields of the electric EDBH are

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\begin{aligned}
ds^2 &= V^{-1}dt^2 - Vd\vec{x}^2\,, \\[2mm]
A &= -\frac{e^{+\kappa\phi_0}}{\kappa\sqrt{2}}V^{-1}\,dt\,, \\[2mm]
e^{-2\kappa\phi} &= e^{-2\kappa\phi_0}V\,,
\end{aligned} \tag{33}
$$

where V is given[6] by an expression similar to Eq. (3):

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[6]As different from the ERN case, dropping the 1 in the expression for V does not give another solution.

$$V(\vec{x}) \;=\; 1 \;+\; \sum_s \frac{2GM_s}{|\vec{x} \;-\; \vec{x}_s|} \,. \tag{34}$$

The *mass* of the sth EDBH is M_s, its electric charge is $Q_s = \sqrt{2}e^{+\kappa\phi_0}G\kappa^{-1}M_s$ and its dilaton charge is $\Sigma_s = -\frac{e^{-2\kappa\phi_0}Q_s^2}{2G\kappa^{-1}M_s} = -G\kappa^{-1}M_s$ and the $d = 4$, $N = 4$ supergravity Bogomol'nyi bound is saturated for each black hole:

$$M_s^2 + \kappa^2 G^{-2}\left(\Sigma_s^2 - e^{-2\kappa\phi_0}Q_s^2\right) = 0\,. \tag{35}$$

We are interested in only establishing the finiteness of the norm of the gravitini and dilatini. The values of these norms will depend on the coefficients these fields appear with in the $N = 4$ supergravity action. Since these numbers will be convention–dependent, we will simply write our expressions in terms of the norms of the constant spinors these zero modes are given in terms of. In particular, the gravitini and dilatini zero modes are:

$$\psi_I \;=\; \frac{1}{\kappa}[\sigma_{0i}V^{-9/4}\partial_{\underline{i}}V\,dt \;-\; 2V^{-5/4}(\partial_{\underline{i}}V - 2\sigma_{ij}\partial_{\underline{j}}V)]\mathcal{C}_I^{(\bar{k})}dx^i \;,$$

$$\lambda_I \;=\; \frac{1}{\kappa}V^{-\frac{7}{4}}\,\gamma_i\partial_{\underline{i}}V\mathcal{C}_I^{(\bar{k})}\,. \tag{36}$$

The norms of these fields are then found to be given by a hypersurface integral times the norms, $\|\mathcal{C}_I^{(\bar{k})}\|^2$, of the constant, Majorana spinors, $\mathcal{C}_I^{(\bar{k})}$. In terms of V, this hypersurface integral is the same as that which appeared in the $N = 2$ case. As we saw above, functionally, the V's differ only in that the mass, M, in the $N = 2$ case is replaced by $2M$ for $N = 4$. Hence the calculation of the $N = 4$ norms follows from the $N = 2$ case. We then find

$$\|\psi_I\|^2 \;=\; M\|\mathcal{C}_I^{(\bar{k})}\|^2 \;, \tag{37}$$

and

$$\|\lambda_I\|^2 \;=\; M\|\mathcal{C}_I^{(\bar{k})}\|^2 \;. \tag{38}$$

Thus both the gravitini and dilatini zero modes are normalizable.

That these fields have finite norm also follows from the Nester theorem for $N = 4$ supergravity [13]. However, explicit values for the individual norms of the gravitini and dilatini cannot be obtained from that construction. As before, we have not used the Witten condition.

We can use S-duality for the evaluation of the norm of the gravitino and dilatino, for axion-dilaton black holes [21] with axion and dilaton fields $a(x)$ and $e^{-2\phi(x)}$, which can be generated

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out of the purely electric ones that we have considered above. According to the discussion of the $N = 2$ case, it is enough to know how the supersymmetry transformation rules of the fermions behave under $SL(2, \mathbf{R})$ transformations[7] [22]

$$(\delta_\epsilon \psi_{I\mu})' = e^{\frac{i}{2}\gamma^5 Arg(S)}\delta_\epsilon \psi_{I\mu} \ , \qquad (\delta_\epsilon \lambda)'_I = e^{\frac{3i}{2}\gamma^5 Arg(S)}\delta_\epsilon \lambda_I \ . \tag{39}$$

where $S \equiv \left[c\left(a(x) + ie^{-2\phi(x)}\right) + d\right]$ and c, d are the elements of the $SL(2, \mathbf{R})$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ . \tag{40}$$

Thus the norm of the gravitino and dilatino which was calculated above is invariant under $SL(2, \mathbf{R})$ symmetry and therefore the result remains valid for the general axion-dilaton black holes of Ref. [21].

4   Holino-hypermultiplet and rigid supersymmetry

With one bound saturated, half of the original supercharges act non-trivially. These generate the spectrum of the resulting system. It is known [12, 5, 23] to be that of the $N = 2$ hypermultiplet. In Ref. [23], we called the massive $M = |Z|$ black hole multiplet of $N = 2$ supersymmetry a holino supermultiplet. The ERN black holes, embedded into $N = 2$ supergravity form the Clifford vacuum for the multiplet with the highest $SU(2)$-spin $J = \frac{1}{2}$. The generic matter multiplet of $N = 2$ supersymmetry is called the hypermultiplet. There was a "mysterious doubling of states" in the spectrum of the hypermultiplet, according to M. Sohnius [24]. Indeed, the spectrum of the hypermultiplet is a doubled version of the massive Wess-Zumino model. However, since the super black hole multiplets have been clearly recognized as forming such multiplets, we now understand this doubling.

All extreme black holes possessing superhair have an intrinsic way of providing a natural doubling of the Clifford vacuum of the corresponding multiplet. In the basis of the supersymmetry algebra in which the central charge is real, there is a degeneracy of the states: for the same value of a mass, the charge of the black hole can take either a positive or a negative value. It is this degeneracy of the black hole solutions which gives an explanation of the "mysterious doubling of states" in the spectrum of states with the mass, equal to the moduli of the central charge of the state.

$$Z = M \qquad \leftarrow \qquad \mathrm{CPT - conjugate} \qquad \rightarrow \qquad Z = -M$$

Using the superhair one can build the black hole supermultiplet; the holino in $N = 2$ supergravity. The original Deser-Teitelboim [25] supercharge of the theory is given in terms of the gravitino as

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[7]Quantum mechanical effects break this symmetry group to $SL(2, Z)$.

$$Q = -i\frac{1}{\kappa}\gamma_5 \oint_{S^2_\infty} \gamma \wedge \psi \quad, \tag{41}$$

where S^2_∞ is the two sphere at spatial infinity. It was specified for the ERN black hole in [12, 5]. Now, expanding the gravitino field in terms of the zero-mode discussed above and the non-zero-modes (which we ignore henceforth), we find that part of Q is now proportional to the creation operator associated to the zero-mode. This part of the supercharge generates the spectrum of the $N = 2$ hypermultiplet.

$$| >_+ \qquad Q_1^\dagger| >_+ \qquad Q_2^\dagger| >_+ \qquad Q_1^\dagger Q_2^\dagger| >_+ \tag{42}$$

$$| >_- \qquad Q_1^\dagger| >_- \qquad Q_2^\dagger| >_- \qquad Q_1^\dagger Q_2^\dagger| >_- \tag{43}$$

The upper (lower) line shows the generation of states with the Clifford vacuum corresponding to the positively (negatively) charged black hole.

Before proceeding, we would like to further illustrate how the supermultiplet of states above arises in the quantization of the spin–$\frac{3}{2}$ field. First we recall that the quantization of the Rarita-Schwinger field yields all of the states (plus parity partners) in the tensor product or Lorentz representations: $(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},0) = (1,\frac{1}{2}) \oplus (0,\frac{1}{2})$. The pure spin–$\frac{1}{2}$ or $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ is projected out by imposing $\gamma^\mu \psi_\mu = 0$. The local supersymmetry provides the gauge parameter for this projection. However, in the ERN background, none of the original supersymmetries survive as local supersymmetries. They act rigidly only. This means that there are no local parameters which can be used to gauge away the pure spin-$\frac{1}{2}$ degrees of freedom. Thus we will be left with a dynamical Dirac spin-$\frac{1}{2}$ field. Using the $N = 2$ rigid supersymmetries, we then conclude that this spin-$\frac{1}{2}$ field is super-partnered with bosonic fields thereby forming the $N = 2$ hypermultiplet.

The hypermultiplet, as given by Sohnius, with *off-shell central charge* describes the same multiplet of states with 2 complex scalars, Dirac spinor and two complex auxiliary fields [24]

$$\Phi_I = (A_I, \psi, F_I) \quad . \tag{44}$$

The underlying quantum field theory, describing the free black hole multiplet is

$$\begin{aligned}
\mathcal{L} &= \frac{i}{2}(\bar{\Phi}^I, \delta_z \Phi_I) + \frac{m}{2}(\bar{\Phi}^I, \Phi_I) \\
&= \frac{1}{2}\partial_\mu A^{I\dagger}\partial^\mu A_I + i\bar{\psi}\,\slashed{\partial}\psi + F^{I\dagger}F_I + m(\tfrac{i}{2}A^{I\dagger}F_I - \tfrac{i}{2}F^{I\dagger}A_I + \bar{\psi}\psi) \quad,
\end{aligned} \tag{45}$$

where the central charge transformation

$$\delta_z \Phi_I = (F_I, \,\slashed{\partial}\psi, \,\Box A_I) \quad, \tag{46}$$

commutes with $N = 2$ global supersymmetry.

The Noether supersymmetry charge derived by quantization of the hypermultiplet Lagrangian will generate the same set of states as the one generated by the black hole superhair. We may conclude therefore that $N = 2$ supergravity in the strong coupling limit may be represented by soliton type states whose own dynamics may be described (before interaction) by the free hypermultiplet action.

In the series of papers by Aichelburg and Embacher [5] about the supergravity solitons, the following conclusion has been reached. The free ERN black hole solitons are described by the relativistic Lagrangian in Eq. (6.10) of the fourth paper in Ref. [5]. This is the massive hypermultiplet Lagrangian (45) with the auxiliary fields F_I excluded by their equations of motion. Besides explaining the hypermultiplet structure of the ERN solitons Aichelburg and Embacher [5] have performed an analysis of the possible interactions which the soliton system may have, in view of the fact that the multi-black hole solutions are also available. They have made an approximation of " slow motion and large distance" to find the possible interactions in the two-soliton system. The resulting picture is the following: there are two types of solitons, with the positive and negative charge. The non-relativistic interaction is given in terms of the Hamiltonian, which is given by three parts. One acting on two-soliton states of both positive charges, the second one acting on the two-soliton state of both negative charges and the last part, acting on the two-soliton state with solitons of opposite charges. The authors suspected that the total picture may be a non-relativistic limit of some covariant field theory. In such a relativistic theory, the particles of the opposite electric charges would become antiparticles of each other.

Our purpose in what follows is to investigate the possible interactions of the ENR black holes which may be described by the full Lorentz-covariant interacting Lagrangian whose free part is the hypermultiplet action with the off-shell central charge (45). It is important to stress that the relativistic action (45) describes both types of soliton states with positive and negative charges and, in this respect, is capable of representing these two types of non-relativistic solitons as antiparticles of each other.

Having seen that the $N = 2$ hypermultiplet arises in the quantization of the zero-mode part of the gravitino, we now wonder if such a multiplet may be placed on the background for which this zero-mode exists. To check this, we must first find rigid parameters. From the structure of the multiplet, we see that we need two such parameters.

Fix the masses of a ERN multi–black hole background. Identify the associated Killing and anti-Killing spinors. For given masses, these spinors are distinguished by different signs of the charges; call these two parameters collectively, ϵ^I. Now place the $N = 2$ massive hypermultiplet on this background. It is important that all derivatives (covariant with respect to this background) may be replaced by $\overline{\nabla}$'s at the expense of a surface term. Consequently, since the ϵ's are constant with respect to $\overline{\nabla} = \hat{\overline{\nabla}}$, the action

$$S_{\text{hyp}}^{N=2} = \int d^4x \ \sqrt{-g} \ \left[\frac{1}{2}g^{\mu\nu}\nabla_\mu A^{I\dagger}\nabla_\nu A_I + i\bar{\psi}\,\overline{\nabla}\,\psi + F^{I\dagger}F_I \right.$$
$$\left. +m(\tfrac{i}{2}A^{I\dagger}F_I - \tfrac{i}{2}F^{I\dagger}A_I + \bar{\psi}\psi)\right] \tag{47}$$

is invariant under the $N = 2$ rigid supersymmetry transformations,

$$\delta A_I \;\; = \;\; 2\bar{\epsilon}_I \psi \; + \; \zeta F_I \;\; ,$$

$$\delta \psi \;\; = \;\; -i\epsilon^I F_I \; - \; i \slashed{\nabla} \epsilon^I A_I \; + \; \zeta \slashed{\nabla} \psi \;\; ,$$

$$\delta F_I \;\; = \;\; 2\bar{\epsilon}_I \slashed{\nabla} \psi \; + \; \zeta \triangle A_I \;\; . \tag{48}$$

The parameters, ϵ_I may be combined to form the Dirac spinor

$$\epsilon(x) \;\; = \;\; V^{-1/2}(x) \, \epsilon_0 \;\; = \;\; \left(1 \; + \; \sum_s \frac{GM_s}{|\vec{x} \; - \; \vec{x}_s|} \right)^{-1/2} \epsilon_0 \;\; , \tag{49}$$

and central charge parameter is

$$\zeta(x) = V^{-1} \zeta_0 = \left(1 \; + \; \sum_s \frac{GM_s}{|\vec{x} \; - \; \vec{x}_s|} \right)^{-1} \zeta_0 \;\; , \tag{50}$$

where ϵ_0, ζ_0 are the values of the global supersymmetry and central charge transformation parameters in the flat background.

The replacement of $\slashed{\nabla}$ by $\hat{\slashed{\nabla}}$ was made so that the (anti-)Killing spinors may be used thereby allowing us to establish these supersymmetries. The rigid $N = 2$ supersymmetry elucidated above is based on the parameter of transformation which solves a massless Dirac equation in the background of the Majumdar–Papapetrou metric. One can show that

$$\epsilon(\vec{x}) \;\; = \;\; V^{-1/2} \epsilon_0 \;\; , \tag{51}$$

are solutions of the Dirac equation

$$\slashed{\nabla} \epsilon \;\; = \;\; 0 \;\; . \tag{52}$$

Here, ϵ_0 is an arbitrary constant Dirac spinor.

The supersymmetry algebra reads

$$\{\bar{Q}^I, Q_J\} \;\; = \;\; i2\delta_J{}^I (\hat{\slashed{\nabla}} \; + \; \mathcal{Z}) \;\; ,$$

$$[\mathcal{Z}, Q_I] \;\; = \;\; 0 \;\; . \tag{53}$$

where Q_I is the supersymmetry charge and $\mathcal{Z}$ is the central charge generator. We note that the parameters which appear on the right-hand-side of the commutator of two supersymmetries are $K_\mu = \bar{\epsilon}_I^{(2)} \gamma_\mu \epsilon^{(1)\,I}$ and $\zeta = \bar{\epsilon}_I^{(2)} \epsilon^{(1)\,I}$. The first is a Killing vector while the second is identified as the central charge parameter.

Thus, the presence of gravitino zero modes and rigid supersymmetry in a certain curved background has led us, following a brief analysis of its quantization, to an action which is that of a novel rigidly supersymmetric theory. This action is presumably the candidate action for the supersymmetric excitations of the ERN black hole.

5 Discussion

We have found that the normalizability of the gravitini zero modes is correlated with the existence of a modified WIN construction in the absence of the source term. The evaluation of the integrals for the norms of the black holes, which we have performed in this paper was consistent with the use of the Gauss theorem with the contribution coming only from the surface at asymptotic infinity. The reason for this was the vanishing of the integrand at the horizon. If the constant-time slices include singularities, this term may contribute. It would be interesting to know what happens in the more complicated case of the supersymmetric but singular IWP configurations of $N = 2$ supergravity [18] and in the case of the supersymmetric IWP configurations of $N = 4$ supergravity (dilaton-axion gravity) [26].

Our results for the finiteness of the gravitino norm in the $3+1$ dimensional MP configurations and axion-dilaton black holes are in contrast with the situation in $2 + 1$ dimensions studied in [10], where the norm was found to be infinite. Additionally, it is interesting that in a closely related $2+1$ dimensional theory it has been recently found [27] that no bound, which is normally derived from the standard WIN construction, exists.

We would like now to compare our calculations of the norm with the calculations of the moduli space of the two-black-hole configurations [28]. Some of the integrals used there resemble the integrals we have found for the norms of the fermion zero modes. In particular, for the black holes considered here, the expression used for the moduli space metric was given in Ref. [28]. For the ERN case ($a = 0$) and for dilaton black holes ($a = 1$) the moduli space metric was calculated from the following integrals:

$$\gamma(r; a) \ \sim \ \int d^3x \ V^{\frac{2(1-a^2)}{(1+a^2)}} \left(|\vec{\partial}V|^2 - \frac{m_1^2}{|\vec{r}_1|^4} - \frac{m_2^2}{|\vec{r}_2|^4} \right) \ , \tag{54}$$

where

$$V = 1 + \frac{m_1}{|\vec{r}_1|} + \frac{m_2}{|\vec{r}_2|} \ , \qquad \vec{\partial}V = \frac{m_1\vec{r}_1}{|\vec{r}_1|^3} + \frac{m_2\vec{r}_2}{|\vec{r}_2|^3} \ , \tag{55}$$

and $\vec{r}_1 = \vec{x} - \vec{x}_1$, $\vec{r}_2 = \vec{x} - \vec{x}_2$, $\vec{r} = \vec{x}_1 - \vec{x}_2$. The crucial difference between these expressions and our expression for the gravitini norm in the two-black hole case

$$\int d^3x \ V^{-2}|\vec{\partial}V|^2 \ , \tag{56}$$

is the pre-factor V^{-2}. For the moduli metric such terms are V^{+2} or V^0. If we take the domain of integration to be $\mathbf{R}^3$, as was done in Ref. [28], near each horizon $V^{-2} \to 0$, however $V^{+2} \to \infty$ and $V^0 \to 1$. We understand therefore that the calculations of the moduli space metric may not be unambiguous and may require additional confirmation. The choice of regularization near the horizon may, under some circumstances, affect the result.

It is therefore quite satisfying that the expressions for the fermion zero-mode norms for all black holes which we have considered in this paper were particularly simple. In particular, if we were to extend the domain of integration to $\mathbf{R}^3$, we would not need to introduce any regularization near the black hole horizons. However, our considerations apply only to supersymmetric black

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holes, whereas the moduli space metric has divergences near the horizon for arbitrary dilaton coupling a. The zero mode calculation which we have performed would not be generalized for arbitrary dilaton coupling. The importance of the finiteness of the norm lies in the fact that this allows us to construct the supersymmetric multiplets including the black hole partners. This presents an alternative possibility to study black hole supersymmetric multiplets and their possible interactions in the framework of relativistic quantum field theory or string theory or perhaps even string field theory, avoiding the non-relativistic approximation.

In the present paper, we have argued that the dynamics of the non-interacting supersymmetric holino multiplet with the bosonic part given by the ERN black hole is described by the free Sohnius hypermultiplet action with an off-shell central charge. The BPS $M = |Z|$ condition is realized only on-shell. We have shown that this theory can be placed in the corresponding gravitational multi-black-hole background with the global supersymmetry of the free theory generalized to the rigid one in the background. Under the condition that the most general interaction of these super-black-hole states preserves the $N = 2$ supersymmetry with the central charge equal to the mass of the multiplet on shell, one can try to describe the interacting ERN black holes in the framework of a relativistic quantum field theory. We expect that such a description would make use of the recent progress in understanding the structure of the superpotential for $N = 2$ supersymmetric sigma models [29].

Acknowledgements

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